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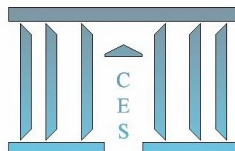
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**Redistribution by Means of Lotteries**

Stéphane GAUTHIER, Guy LAROQUE

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# Redistribution by Means of Lotteries\*

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## Abstract

A government designs transfers to agents in the absence of information on their preferences. The second-best allocation is equal sharing among citizens when the awards are deterministic. We provide a necessary and sufficient condition under which lotteries improve upon the egalitarian outcome. The condition requires that the citizens with large social weights have low risk aversion, and that the left tail of the distribution of risk aversion be sufficiently dispersed.

**JEL classification numbers:** H21, H23, H26.

**Keywords:** Lerner egalitarianism, random redistribution, incentives, qualified constraints.

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# 1 Introduction

Premium Bonds have been introduced in 1956 by then Chancellor of the Exchequer Harold Macmillan to promote saving in the UK. These bonds combine a riskless principal with additional prizes paid randomly. Every £1 face value bond has a small equal chance of winning a prize ranging from £25 to £1m each month. Holders can cash their bonds at their £1 face value at any time. The lottery is very popular in the UK: around 23 million individuals hold Premium Bonds, and it is particularly appealing to low income families (Tufano (2008)). Similar lottery-linked saving programs also exist in other developed and developing countries (Guillén and Tschoegl (2002)). The aim of our paper is to discuss the efficiency and equity of such programs. Low income families benefit from non negative expected income transfers but they also suffer from randomness, all the more as they are risk averse. At first sight welfare losses seem useless and they could be avoided by removing the noise component, e.g., by appealing to the Income Bonds that closely resemble Premium Bonds but pay the risk-free interest. There is an extensive literature which discusses efficiency and equity of lotteries and other forms of gambling as means of raising funds, e.g., raffles, pari-mutuel raffles or lotto. Lotteries are usually found dominated by other (deterministic) tax instruments. Morgan (2000) provides a synthesis of this literature and advocates lotteries in the case of public good provision by private charities lacking tax power. The particular feature of the Premium Bonds lottery is to involve non negative gain whatever the outcome of the lottery.

Similar issues arise in other circumstances, e.g., the organization of conscription discussed in (Sabin (2008)) or the allocation of health care treatments. The conscription can indeed be combined with a commutation tax, or with the possibility given to the young men called for duty to find a substitute. The wealthier citizens then pay to avoid the risk associated with war which are born by the less well-off part of the population. As in the case of Premium Bonds, the outcome is to transfer some income toward the poorest and/or less risk averse individuals who are exposed to risk. An health agency may face a similar problem when allocating new medical treatments that involve more risk but higher gains in life expectancy.

In these examples an authority chooses to expose a part of the population to a situation which entails more risk but higher expected gains. It is known that in a second-best world a principal may find it valuable to design a random contract to the agents (Laffont and Martimort (2002)). Risk sometimes relaxes the incentive constraints when second-best considerations result from asymmetric information. Typically one expects randomization to be rare, and the literature provides many context-specific conditions ensuring that a deterministic optimum cannot be improved upon using randomization. However little is known about general conditions for useful/useless randomization. In Gauthier and Laroque (2014) we give a necessary and sufficient condition for local randomization around a deterministic optimum to yield no welfare improvement, but our previous analysis only applies to well-behaved problems where the constraints in the deterministic program are qualified.

In the examples discussed above, the qualification requirement fails to hold: if the

government is committed to use only deterministic programs, and if it cannot observe the actual saving capacity of the household, the actual opportunity cost of conscription, or the actual health status, then every agent will pretend to have the characteristics which yield the unique best outcome. Only the risk-free bonds will be bought, and no additional saving will be made. All the young men will try to avoid war by pretending to have high opportunity costs associated with conscription, and patients will ask for the sure treatment giving the highest gain in life expectancy. In the absence of noise, no screening is possible and all the agents have to be treated symmetrically. Such situations are particular instances of a general redistribution problem identified by Lerner (1944). Consider a government trying to allocate in a deterministic fashion a given sum of money, and potential income recipients with different income valuations. If valuation is not observed by the government, and recipients always value more income than less income, then only an equal sharing of the resources is feasible.

A random allocation may allow the planner to sort individuals by their attitudes toward risk. Randomization induces individuals to reveal private information that can be used by the planner to achieve better outcomes, e.g., to distribute more resources to those who have a larger social weight. Indeed Pestieau, Posen, and Slutsky (2002) present a number of examples of economies where this phenomenon occurs. Some of these examples involve large departures from the Lerner egalitarian allocation, others are restricted to small deviations from the Lerner outcome. Following the latter, our paper studies the possibility of a second best improving local randomization.

We provide a necessary and sufficient condition for the existence of random locally improving stochastic allocations in the Lerner setup. As in the work of Pestieau, Posen, and Slutsky (2002), this condition requires that society favors individuals with a low risk aversion: it is then possible to transfer to these agents the average tax obtained from high risk aversion individuals. Such transfers must be allocated randomly so that the individuals with a low social value do not apply. Our necessary and sufficient condition generalizes this basic insight and shows that random redistribution is socially useful when the average social value of the high risk aversion individuals is low and the dispersion of risk aversion at the bottom of the distribution is high enough.

This result calls for creating lotteries with positive expected gain when there is a negative correlation between social weights and risk aversions. Individuals with a high risk aversion would pay for not entering these lotteries. The experimental results reported in Lobe and Hölzl (2007) suggest that low income families involved in the Premium Bonds lottery indeed have lower risk aversion than the richest.

## 2 Deterministic redistribution

A social planner has to allocate a total fixed income  $\bar{y}$  to individuals who differ in their utilities for income. There is a continuum of individuals with total unit mass. The utility of a type  $\theta$  individual is  $u(y, \theta)$  when her income is  $y$ ,  $y \in \mathbb{R}^+$ . The parameter  $\theta$  is distributed on a closed rectangle  $\Theta$  of a finite dimensional space, with a positive continuous probability

density function  $f(\cdot)$ , and cumulative distribution function  $F(\cdot)$ . Utility is twice continuously differentiable, increasing and concave in income:  $u'_y(y, \theta) > 0$  and  $u''_{yy}(y, \theta) < 0$  for all  $y$  and  $\theta$ .

Each individual is supposed to know her own type. In the first-best the social planner also observes this type and allocates  $y^*(\theta)$  to each type  $\theta$  individual. The profile  $(y^*(\theta))$  maximizes<sup>1</sup>

$$\int_{\Theta} u(y(\theta), \theta) dF(\theta) \quad (1)$$

subject to the feasibility constraint

$$\int_{\Theta} y(\theta) dF(\theta) \leq \bar{y}. \quad (2)$$

The first-best optimum  $(y^*(\theta))$  is characterized by (2) at equality and the first-order conditions  $u'_y(y^*(\theta), \theta) = u'_y(y^*(\hat{\theta}), \hat{\theta})$  for all  $\theta$  and  $\hat{\theta}$ . In general heterogeneity in utility implies an unequal division of income.

The implementation of this solution crucially requires that the planner observes agents' types. As Lerner (1944) acknowledges, in the absence of public information on types,

‘[ $\dots$ ] every individual could declare that he has exceptionally high capacities for satisfaction and so should be given more income than anybody else if total satisfaction is to be maximized. [ $\dots$ ] If it is impossible, on any division of income, to discover which of any two individuals has a higher marginal utility of income, the probable value of total satisfactions is maximized by dividing income evenly.’ (Lerner (1944), chapter 3, page 28-29)

Indeed, when the social planner no longer observes the individual types, the optimal income distribution maximizes (1) subject to (2) and the incentive constraints

$$u(y(\theta), \theta) \geq u(y(\hat{\theta}), \theta) \text{ for all } \theta \text{ and } \hat{\theta}.$$

Since utility is increasing in income, these constraints are satisfied if and only if  $y(\theta) = y(\hat{\theta})$  for all  $\theta$  and  $\hat{\theta}$ . The feasibility constraint then gives the only allocation consistent with all the constraints,

$$y(\theta) = \bar{y} \text{ for all } \theta. \quad (3)$$

In a Lerner economy, incentive constraints yield income equalization independently of the social redistributive tastes. This is the Lerner (deterministic second-best) optimum.

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<sup>1</sup>To alleviate notations, we write

for

$$\int_{\Theta} dF(\theta) = \int_{\theta_1^{\inf}}^{\theta_1^{\sup}} \cdots \int_{\theta_n^{\inf}}^{\theta_n^{\sup}} f(\theta) d\theta_1 \cdots d\theta_n.$$

### 3 Random redistribution

We now extend the powers of the planner and allow for randomized redistribution. Suppose that the planner can design a menu of lotteries, such that every individual must choose a lottery from the menu. Given the random (positive or negative) draw from the lottery, there is commitment both from the government and the players to conform to the outcome. Suppose also that a law of large number holds, so that with independent draws the cost of the lottery in feasibility terms is equal to its mathematical expectation. Then an astute design of the set of available lotteries may lead the members of the society *ex ante* to reveal their types.

We look for a menu of lotteries ( $\tilde{y}(\theta)$ ) that improves upon the reference Lerner optimum,

$$\int_{\Theta} \mathbb{E}[u(\bar{y} + \tilde{y}(\theta), \theta)] dF(\theta) > \int_{\Theta} u(\bar{y}, \theta) dF(\theta) \quad (4)$$

subject to the feasibility constraint

$$\int_{\Theta} \mathbb{E}[\tilde{y}(\theta)] dF(\theta) = 0, \quad (5)$$

and the incentive constraints

$$\mathbb{E}[u(\bar{y} + \tilde{y}(\theta), \theta)] \geq \mathbb{E}[u(\bar{y} + \tilde{y}(\hat{\theta}), \theta)] \text{ for all } \theta \text{ and } \hat{\theta}. \quad (6)$$

The lotteries  $\tilde{y}(\theta)$  that we shall consider are small, meaning that their support are contained in a ball around the origin. The mean and the variance of a typical lottery are  $m \equiv \mathbb{E}[\tilde{y}]$  and  $v \equiv \text{var}[\tilde{y}]$ . In the Lerner optimum,  $\tilde{y}(\theta) = m(\theta) = v(\theta) = 0$  for all  $\theta$ . Let

$$r(\theta) = -\frac{u''_{yy}(\bar{y}, \theta)}{u'_y(\bar{y}, \theta)}$$

be the coefficient of absolute risk aversion of a type  $\theta$  individual evaluated at the Lerner outcome. Also let  $\alpha(\theta) \equiv u'_y(\bar{y}, \theta)$  denote the marginal social weight of a type  $\theta$  individual. In the sequel, the weights are normalized so that their total sum in the population is 1,

$$\int_{\Theta} \alpha(\theta) dF(\theta) = 1.$$

**Lemma 1.** *Consider a family of menus  $\tilde{y}_\lambda(\theta)$ ,  $\theta \in \Theta$ , with mean variance  $(\lambda m(\theta), \lambda v(\theta))$ , for  $\lambda$  in the interval  $[0, 1]$ . Suppose that they satisfy the incentive compatibility constraints (6). Consider two types  $\theta_1$  and  $\theta_2$  with  $r(\theta_1) < r(\theta_2)$ . For  $\lambda$  small enough, the incentive constraints imply*

$$v(\theta_1) \geq v(\theta_2),$$

$$m(\theta_1) \geq m(\theta_2)$$

and

$$m(\theta_1) - \frac{r(\theta_1)}{2}v(\theta_1) \geq m(\theta_2) - \frac{r(\theta_2)}{2}v(\theta_2).$$

If  $m$  and  $v$  are continuously differentiable with respect to some component  $t$  of  $\theta$  in a neighbourhood of  $\theta_1$ , then

$$\frac{\partial m(\theta_1)}{\partial t} = \frac{r(\theta_1)}{2} \frac{\partial v(\theta_1)}{\partial t}.$$

*Proof.* A second order development of individual utility yields

$$\mathbb{E}[u(\bar{y} + \tilde{y}_\lambda(\theta), \theta)] = u(\bar{y}, \theta) + u'_y(\bar{y}, \theta)\lambda m(\theta) + \frac{1}{2}u''_{yy}(\bar{y}, \theta) [(\lambda m(\theta))^2 + \lambda v(\theta)] + \mathbb{E}[\lambda^2 o_\theta(\tilde{y}_\theta^2)],$$

which can be rewritten as

$$\mathbb{E}[u(\bar{y} + \tilde{y}_\lambda(\theta), \theta)] = u(\bar{y}, \theta) + u'_y(\bar{y}, \theta)\lambda \left[ m(\theta) - \frac{r(\theta)}{2} (\lambda m(\theta)^2 + v(\theta)) \right] + \mathbb{E}[\lambda^2 o_\theta(\tilde{y}_\theta^2)].$$

The incentive constraint of the  $\theta_1$  type then is

$$m(\theta_1) - m(\theta_2) \geq \frac{r(\theta_1)}{2}[\lambda m(\theta_1)^2 + v(\theta_1) - \lambda m(\theta_2)^2 - v(\theta_2)] + \mathbb{E}[\lambda^2 o(\tilde{y}^2)]. \quad (7)$$

Adding up with that of the  $\theta_2$  individual yields

$$0 \geq (r(\theta_1) - r(\theta_2)) [\lambda m(\theta_1)^2 + v(\theta_1) - \lambda m(\theta_2)^2 - v(\theta_2)] + \mathbb{E}[\lambda o(\tilde{y}^2)].$$

Letting  $\lambda$  tend towards zero gives the first inequality. Again letting  $\lambda$  tend to zero in (7) implies  $m(\theta_1) \geq m(\theta_2)$  and proves the second inequality of the lemma. Finally, to prove the third inequality, note that (7) at the limit when  $\lambda$  tends to zero,

$$m(\theta_1) - \frac{r(\theta_1)}{2}v(\theta_1) \geq m(\theta_2) - \frac{r(\theta_2)}{2}v(\theta_2),$$

gives

$$m(\theta_1) - \frac{r(\theta_1)}{2}v(\theta_1) \geq m(\theta_2) - \frac{r(\theta_2)}{2}v(\theta_2)$$

since  $r(\theta_1) < r(\theta_2)$ .

Finally, to prove the final assertion of the Lemma, fix  $t_1$ , take two sequences of  $t_2$  converging to  $t_1$ , one from above, the other from below, and get the derivatives in the limit by dividing through by  $t_2 - t_1$ . ■

The incentive constraints imply that the types who display a higher risk aversion at the Lerner outcome face lotteries with both lower mean and lower variance, and also get



a lower expected utility. Any redistribution of welfare toward risk averse individuals by means of lotteries is not incentive compatible.

From Lemma 1, it follows that to any menu satisfying (4), (5) and (6) one can associate an improving menu such that the allocation of the individual with the largest risk aversion  $r^{\sup} \equiv \max r(\theta)$  is non random: one can just reduce uniformly the variances of all the lotteries in the original menu by  $v(r^{\sup})$ .

While the Lerner program and its formalization are rather specific, there are a number of analogous situations where random contracts and selection according to risk aversion may be used by the government. Below we sketch the three examples discussed in Introduction.

*Example 1.* A State is involved in a war and needs to select a given number  $n$  of citizens who will fight. Conscription is associated with a commutation tax. Citizens differ according to their incomes and their attitudes towards military enrollment summarized by  $(y, \theta)$ , which are private information. The State proposes two options. The first is to go to war. Soldiers are exposed to the risk of fighting with its associated lottery of monetary compensations  $\tilde{y}_0$ . The second option is to avoid the risk of fighting while paying for the war a deterministic income  $y_1$ . A citizen  $(y, \theta)$  chooses to go to war if and only if

$$\mathbb{E}[u(y + \tilde{y}_0, \theta)] \geq u(y - y_1, \theta).$$

Let  $W$  be the set of  $(y, \theta)$  such that this inequality is satisfied, with measure  $F(W)$ . The reform is feasible if it satisfies the budget constraint

$$F(W)\mathbb{E}[\tilde{y}_0] - (1 - F(W))y_1 \leq 0.$$

The contribution to welfare is

$$\int_W \mathbb{E}[u(y + \tilde{y}_0, \theta)] dF(y, \theta) + \int_{\bar{W}} \mathbb{E}[u(y - y_1, \theta)] dF(y, \theta) > 0.$$

In addition the army must be large enough,  $F(W) \geq n$ . ■

*Example 2.* The problem is to allocate health treatments. The outcome is life expectancy, measured by  $y$ . The parameter  $\theta$  can be thought of as an initial health status. The health agency proposes a collection of treatments. Treatment  $t$  has a random impact on life expectancy measured by  $\tilde{y}(t)$ . New advanced treatments yield a higher increase in the expected life expectancy, but greater uncertainty relatively to some benchmark widely used treatment. The expected utility of a type  $\theta$  patient when she receives treatment  $t$  is  $\mathbb{E}[u(\tilde{y}(t), \theta)]$ . The ‘small noise’ case considered in Lemma 1 can be viewed as a situation where the new treatments are small incremental innovations with respect to the benchmark. The government chooses a profile of treatments  $(t(\theta))$ . Incentive constraints require that

$$\mathbb{E}[u(\tilde{y}(t(\theta)), \theta)] \geq \mathbb{E}[u(\tilde{y}(t(\hat{\theta})), \theta)]$$

for all  $\theta$  and  $\hat{\theta}$ . From the government viewpoint, the individual utility is scaled by a factor  $\alpha(\theta)$ , so that the second best program looks for  $(t(\theta))$  that maximizes

$$\int_{\Theta} \alpha(\theta) \mathbb{E}[u(\tilde{y}(t(\theta)), \theta)] \, dF(\theta)$$

subject to a budget constraint and possibly technological constraints. ■

*Example 3.* The practical situation that looks the closest to our setup is the UK institution of the Premium Bonds which involve non negative random income transfers. Every bond has a face value of one pound. They are not tradable. A fixed number of prizes are available, two monthly prizes of one million pounds and many smaller prizes. The prizes are randomly allocated by a lottery. Let  $\tilde{\rho}(n)$  be the return when the owner has  $n$  bonds,  $n > 0$ . The expected utility of a trader with other (privately known) saving  $s$  yielding the riskless interest  $\rho$ , is  $Eu(\rho(s - n) + n\tilde{\rho}(n))$ . Let  $n(s)$  be the number of premium bonds designed for an individual with saving  $s$ ,  $s \in \mathbb{R}$ , the budget of the National Savings and Investments administration writes

$$\int_{s \in \mathbb{R}} (\rho - E\tilde{\rho}(n(s))) n(s) \, dF(s) \geq 0.$$

This coincides with our setup by setting  $\tilde{y}(s) = (\rho - \tilde{\rho}(n(s))) n(s)$ . ■

## 4 A simple reform

This section provides a sufficient condition for random redistribution to locally dominate the Lerner optimum. The menu is made of only two lotteries. The first  $\tilde{y}$  has mathematical expectation  $\underline{m}$  and variance  $\underline{v}$ , a strictly positive number. The second in fact is certain, with mathematical expectation  $\overline{m}$ . The agents who prefer the first lottery have a type  $\theta$  in

$$\tilde{\Theta} = \{\theta \mid \mathbb{E}[u(\underline{y} + \tilde{y}, \theta)] \geq u(\underline{y} + \overline{m}, \theta)\}.$$

For the small lotteries considered in Lemma 1 we have

$$\mathbb{E}[u(\underline{y} + \tilde{y}, \theta) - u(\underline{y}, \theta)] \simeq \alpha(\theta) \left[ \underline{m} - \frac{r(\theta)}{2} \underline{v} \right]$$

and

$$u(\underline{y} + \overline{m}, \theta) - u(\underline{y}, \theta) \simeq \alpha(\theta) \overline{m}.$$

Therefore the set  $\tilde{\Theta}$  actually comprises the types  $\theta$  such that  $r(\theta) \leq r^*$  where

$$\overline{m} = \underline{m} - \frac{r^*}{2} \underline{v}. \tag{8}$$

Let  $G(r)$  be the proportion of agents with risk aversion less than  $r$  at the Lerner optimum. With a slight abuse of notation, let also  $\alpha(r)$  be such that

$$\int_{\theta|r(\theta)=r} \alpha(\theta) dF(\theta) = \alpha(r) dG(r).$$

The change in social welfare associated with this menu of lotteries

$$\int_{\tilde{\Theta}} \alpha(\theta) \mathbb{E}[u(\bar{y} + \tilde{y}, \theta)] dF(\theta) + \int_{\Theta \setminus \tilde{\Theta}} \alpha(\theta) u(\bar{y} + \bar{m}, \theta) dF(\theta) - \int_{\Theta} \alpha(\theta) u(\bar{y}, \theta) dF(\theta) \quad (9)$$

is approximately

$$\int_{r^{\inf}}^{r^*} \alpha(r) \left( \underline{m} - \frac{r}{2} \underline{v} \right) dG(r) + \bar{m} \int_{r^*}^{r^{\sup}} \alpha(r) dG(r).$$

Using the feasibility constraint,

$$G(r^*) \underline{m} + [1 - G(r^*)] \bar{m} = 0,$$

and the definition of the threshold  $r^*$  given in (8) we get

$$\bar{m} = -G(r^*) \frac{r^*}{2} \underline{v} \quad \text{and} \quad \underline{m} = [1 - G(r^*)] \frac{r^*}{2} \underline{v}.$$

Therefore the approximate change in social welfare brought by the menu of lotteries becomes

$$\frac{1}{2} \left[ \int_{r^{\inf}}^{r^*} \alpha(r) ([1 - G(r^*)] r^* - r) dG(r) - G(r^*) r^* \int_{r^*}^{r^{\sup}} \alpha(r) dG(r) \right] \underline{v}.$$

Since  $\underline{v} > 0$ , a sufficient condition for a local improvement upon the Lerner optimum is that the expression that multiplies  $\underline{v}$  be positive for some  $r^*$ , i.e.,

$$r^* \int_{r^{\inf}}^{r^*} \alpha(r) dG(r) - r^* G(r^*) > \int_{r^{\inf}}^{r^*} \alpha(r) r dG(r).$$

Since this inequality cannot be satisfied when  $r^* G(r^*) = 0$ , we have

**Lemma 2.** *A sufficient condition for a random redistribution of income to improve upon the Lerner deterministic outcome is that there is some  $r^* > 0$ ,  $G(r^*) > 0$ , such that*

$$\int_{r^{\inf}}^{r^*} \alpha(r) \frac{dG(r)}{G(r^*)} - 1 > \int_{r^{\inf}}^{r^*} \alpha(r) \frac{r}{r^*} \frac{dG(r)}{G(r^*)}. \quad (10)$$

The left-hand side of (10) is the difference between the average social value of giving one unit of money to each individual whose risk aversion is less than  $r^*$  and the average social value of such a gift to every one (the size of the population is normalized to 1). This difference is positive when individuals with risk aversion smaller than  $r^*$  have a higher social value than the others.

The right-hand side of (10) depends on the shape of the left tail of the risk aversion distribution. The social welfare loss from randomness comes entirely from utility losses of the less risk averse individuals, who support the risk. These losses are proportional to their risk aversions  $r$ ,  $r \leq r^*$ , and weighted by their social values.

When all the individuals with a risk aversion less than  $r^*$  are risk neutral ( $r = 0$  for  $r < r^*$ ), the right-hand side of (10) is equal to 0, so that randomization is useful whenever the social weight of the risk neutral agents is larger than 1.

*Remark 1* (Background risk). Lemma 2 extends to the case where the agents are endowed with an initial random income  $\bar{y}(\theta)$ , instead of the deterministic Lerner outcome  $\bar{y}$ . The government proposes a menu of lotteries  $\tilde{y}(\theta)$  before incomes are realized. Suppose that the lotteries are constrained to be independent of the initial income  $\bar{y}(\theta)$ . Formula (10) then is valid with

$$r(\theta) = -\frac{\mathbb{E}u''(\bar{y}(\theta), \theta)}{\mathbb{E}u'(\bar{y}(\theta), \theta)}, \quad \alpha(\theta) = \mathbb{E}u'(\bar{y}(\theta), \theta).$$

A necessary condition for a random income distribution  $\bar{y}(\theta)$  to be globally optimal is therefore that (10) be *not* satisfied for this menu.

## 5 A necessary condition

Although (10) has been obtained from a special reform with individuals bunching on two lotteries only, this section shows that it is also necessary for an improvement. This yields the main result of our paper.

**Proposition 1.** *Suppose that the small profile  $(\tilde{y}(\theta))$  yields a choice of lotteries which only depends on  $r$  and satisfies the monotonicity properties of Lemma 1. Then there exists  $r^*$ , with  $G(r^*) > 0$ , such that (10) is a necessary and sufficient condition for  $(\tilde{y}(\theta))$  to increase social welfare upon the Lerner optimum.*

*Proof.* We already know that (10) is a sufficient condition for  $(\tilde{y}(\theta))$  to increase social welfare upon the Lerner optimum. It remains to show that it is also necessary. By Lemma 1,  $m(r)$  and  $v(r)$  are monotonic functions. They are therefore differentiable in  $r$  almost everywhere, and integrable on  $[r^{\inf}, r^{\sup}]$ . From the last statement of Lemma 1, we see that at any point where both  $m$  and  $v$  are differentiable functions of  $r$  incentive constraints require that  $m'(r) - rv'(r)/2 = 0$ . Integrating this first-order condition yields

$$m(r) = m(r^{\inf}) + \int_{r^{\inf}}^r \frac{z}{2} v'(z) dz.$$

From the feasibility constraint, one gets

$$m(r^{\text{inf}}) = - \int_{r^{\text{inf}}}^{r^{\text{sup}}} \left( \int_{r^{\text{inf}}}^r \frac{z}{2} v'(z) dz \right) dG(r) = \int_{r^{\text{inf}}}^{r^{\text{sup}}} [1 - G(z)] \frac{z}{2} v'(z) dz.$$

Hence,

$$m(r) = \int_{r^{\text{inf}}}^r \frac{z}{2} v'(z) dz - \int_{r^{\text{inf}}}^{r^{\text{sup}}} [1 - G(z)] \frac{z}{2} v'(z) dz.$$

Reintroducing this expression for the profile ( $m(r)$ ) into the change in the social objective (9), one gets at the first-order in  $\lambda$  following the proof of Lemma 1

$$\int_{r^{\text{inf}}}^{r^{\text{sup}}} \alpha(r) \left( \int_{r^{\text{inf}}}^r \frac{z}{2} v'(z) dz \right) dG(r) - \int_{r^{\text{inf}}}^{r^{\text{sup}}} [1 - G(z)] \frac{z}{2} v'(z) dz - \int_{r^{\text{inf}}}^{r^{\text{sup}}} \alpha(r) \frac{r}{2} v(r) dG(r).$$

Integration by parts yields

$$\int_{r^{\text{inf}}}^{r^{\text{sup}}} \alpha(r) \left( \int_{r^{\text{inf}}}^r \frac{z}{2} v'(z) dz \right) dG(r) = \int_{r^{\text{inf}}}^{r^{\text{sup}}} \left( 1 - \int_{r^{\text{inf}}}^r \alpha(z) dG(z) \right) \frac{r}{2} v'(r) dr$$

and, using  $v(r^{\text{sup}}) = 0$ ,

$$\int_{r^{\text{inf}}}^{r^{\text{sup}}} \alpha(r) \frac{r}{2} v(r) dG(r) = - \int_{r^{\text{inf}}}^{r^{\text{sup}}} \left( \int_{r^{\text{inf}}}^r \alpha(z) \frac{z}{2} dG(z) \right) v'(r) dr.$$

Therefore the change in the social objective rewrites

$$\int_{r^{\text{inf}}}^{r^{\text{sup}}} \left[ \frac{r}{2} \left( G(r) - \int_{r^{\text{inf}}}^r \alpha(z) dG(z) \right) + \int_{r^{\text{inf}}}^r \alpha(z) \frac{z}{2} dG(z) \right] v'(r) dr.$$

By Lemma 1 incentive constraints imply  $v'(r) \leq 0$  for almost all  $r \in [r^{\text{inf}}, r^{\text{sup}}]$ . If there is a profile  $(m(r), v(r))$  improving upon the Lerner optimum, it must be that

$$\left( G(r) - \int_{r^{\text{inf}}}^r \alpha(z) dG(z) \right) \frac{r}{2} + \int_{r^{\text{inf}}}^r \alpha(z) \frac{z}{2} dG(z) < 0$$

for some  $r$ . For  $r$  such that  $G(r) = 0$ , the above inequality cannot be satisfied. It follows that there is some  $r$  such that

$$\left( 1 - \int_{r^{\text{inf}}}^r \alpha(z) \frac{dG(z)}{G(r)} \right) \frac{r}{2} + \int_{r^{\text{inf}}}^r \alpha(z) \frac{z}{2} \frac{dG(z)}{G(r)} < 0.$$

This is (10) with  $r = r^*$ . ■

The work that is the closest to ours in the literature is that of Pestieau, Possen, and Slutsky (2002), which addresses the same broad concern: when is it possible to improve on the Lerner optimum with randomization? They find in their Theorem 3A that for local randomization to be socially useful in a two-type economy, the social planner must put a higher weight on the type with the least risk aversion.<sup>2</sup> Their argument proceeds as follows, supposing for simplicity an equal number of agents of each type. Assume that there exist two lotteries  $\tilde{y}(\theta_1)$  and  $\tilde{y}(\theta_2)$  satisfying feasibility and incentive constraints. Feasibility implies that  $m(\theta_1) = -m(\theta_2)$ . When the support of each lottery is close to 0 a first-order Taylor expansion shows that the social objective is higher than in the Lerner optimum when  $\alpha(\theta_1)m(\theta_1) + \alpha(\theta_2)m(\theta_2) > 0$ . The incentive constraints imply  $m(\theta_1) > 0$  and yield

$$\alpha(\theta_1) - \alpha(\theta_2) > 0.$$

This is to be compared with our expression for (10), which in this setup takes the form

$$\alpha(\theta_1) - \alpha(\theta_2) \geq 2\alpha(\theta_1) \frac{r(\theta_1)}{r(\theta_2)}.$$

Since the right hand side of the above inequality is positive, Proposition 1 confirms the validity of Pestieau, Possen, and Slutsky (2002). Our method of proof allows for an explicit treatment of the agents risk aversion and yields a more precise bound, giving a necessary and sufficient condition for useful local randomization of income transfers.

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<sup>2</sup>Pestieau, Possen, and Slutsky (2002) do not restrict their attention to a neighborhood of the Lerner outcome but also study non local changes. When there is an outcome that is very bad for the agent with low social weight and not so bad for the type whom the planner wants to favor, they show how to build socially improving mixtures using this outcome.

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